

Quantum Entanglement of Moving Bodies

Robert M. Gingrich and Christoph Adami

Quantum Computing Technologies Group, Jet Propulsion Laboratory 126-347, California Institute of Technology, Pasadena, California 91109-8099

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We study the properties of quantum entanglement in moving frames, and show that, because spin and momentum become mixed when viewed by a moving observer, the entanglement between the spins of a pair of particles is not invariant. We give an example of a pair, fully spin entangled in the rest frame, which has its spin entanglement reduced in all other frames. Similarly, we show that there are pairs whose spin entanglement increases from zero to maximal entanglement when boosted. While spin and momentum entanglement separately are not Lorentz invariant, the joint entanglement of the wave function is.

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Mostly, theories in physics are born out of necessity, but not always. The thermodynamics of moving bodies, for example (relativistic thermodynamics), was a hotly contested topic without resolution [1–4] (but see [5]) mostly because no experiment required it. As a side effect, it was learned that the temperature concept in relativistic thermodynamics is ambiguous simply because radiation that is perfectly blackbody in an inertial frame is *not* thermal if viewed from a moving frame [5,6]. This is an interesting result for information theory [7], however, since if probability distributions can depend on the inertial frame, then so can Shannon entropy and information. Even more interesting are the consequences for quantum information theory, where quantum entanglement plays the role of the primary resource in quantum computation and communication [8]. Relativistic quantum information theory may become a necessary theory in the near future, with possible applications to quantum teleportation [9], entanglement-enhanced communication [10], quantum clock synchronization, and quantum-enhanced global positioning [11].

Entanglement is a property unique to quantum systems. Two systems (microscopic particles or even macroscopic bodies [12]) are said to be quantum entangled if they are described by a joint wave function that cannot be written as a product of wave functions of each of the subsystems (or, for mixed states, if a density matrix cannot be written as a weighted sum of product density matrices). The subsystems can be said *not* to have a state of their own, even though they may be arbitrarily far apart. The entanglement produces correlations between the subsystems that go beyond what is classically possible [13]. It is this feature that enables quantum communication protocols such as teleportation and superdense coding. However, the preparation, sharing, and purification of entanglement is usually a complicated and expensive process that requires great care. It is therefore of some importance to understand all those processes that might affect quantum entanglement (in particular, those processes that lead

to decoherence). It was shown recently that Lorentz boosts can affect the marginal entropy of a single quantum spin [14]. Here, we determine that the entanglement between *two* systems depends on the frame in which this entanglement is measured. We show that a fully entangled spin-1/2 system (a Bell state) loses entanglement if observed by a Lorentz-boosted observer. Thus, Lorentz boosts introduce a transfer of entanglement *between* degrees of freedom, that could be used for entanglement manipulation. While the entanglement between spin or momentum alone may change due to Lorentz boosts, the entanglement of the *entire* wave function (spin and momentum) is invariant.

In order to define the momentum eigenstates for a massive particle with spin, we start by defining the rest frame eigenstates,

$$P^\mu |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle p_0^\mu, \quad (1)$$

$$\mathbf{J}^2 |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle s(s+1), \quad (2)$$

$$J_z |\mathbf{0}\lambda\rangle = |\mathbf{0}\lambda\rangle \lambda, \quad (3)$$

where $p_0^\mu = (m, \mathbf{0})$, s is the total angular momentum of the particle, and λ is the z component of angular momentum. Since the particle is at rest, s and λ are the spin and the z component of the spin for the particle, respectively.

We define a momentum state by acting on the rest frame state with a pure Lorentz boost

$$|\mathbf{p}\lambda\rangle \equiv L(\boldsymbol{\xi}_\mathbf{p}) |\mathbf{0}\lambda\rangle, \quad (4)$$

where $L(\boldsymbol{\xi}_\mathbf{p})$ is a boost such that

$$L(\boldsymbol{\xi}_\mathbf{p})(m, \mathbf{0}) = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}), \quad (5)$$

where the rapidity, $\boldsymbol{\xi}_\mathbf{p}$, is given by

$$\sinh|\boldsymbol{\xi}_\mathbf{p}| = \frac{|\mathbf{p}|}{m} \quad (6)$$

$$\frac{\boldsymbol{\xi}_\mathbf{p}}{|\boldsymbol{\xi}_\mathbf{p}|} = \frac{\mathbf{p}}{|\mathbf{p}|}. \quad (7)$$

In what follows, we use \mathbf{p} to represent the 4-vector $(\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$ unless it is ambiguous.

The effect of an arbitrary Lorentz transformation Λ (rotation and boost) on a momentum eigenstate is

$$\Lambda|\mathbf{p}\lambda\rangle = \Lambda L(\xi_{\mathbf{p}})|\mathbf{0}\lambda\rangle \quad (8)$$

$$= L(\xi_{\Lambda\mathbf{p}})L(\xi_{\Lambda\mathbf{p}})^{-1}\Lambda L(\xi_{\mathbf{p}})|\mathbf{0}\lambda\rangle. \quad (9)$$

Since $L(\xi_{\Lambda\mathbf{p}})^{-1}\Lambda L(\xi_{\mathbf{p}})$ leaves $\mathbf{0}$ invariant, it must be a rotation. These rotations are called the Wigner rotations $R(\Lambda, \mathbf{p})$, and they act only on the rest frame spin component λ . Hence, we can write

$$\Lambda|\mathbf{p}\lambda\rangle = \sum_{\lambda'} |\Lambda\mathbf{p}\lambda'\rangle D_{\lambda',\lambda}^{(s)}(R(\Lambda, \mathbf{p})), \quad (10)$$

where $D_{\lambda',\lambda}^{(s)}(R)$ is the spin s representation of the rotation R . Here, we restrict ourselves to $s = 1/2$, but the generalization to larger spins is straightforward. For a review of momentum eigenstates and spin, see [15]. Since a local unitary transformation will not affect any measure of entanglement [16,17], the unitary transformation Λ on the infinite dimensional space of momentum and rest frame spin will not change the entanglement between two particles, provided we do not trace out a part of the wave function. However, in looking at the entanglement between spins, tracing out over the momentum is implied.

The wave function for two massive spin-1/2 particles can be written as

$$|\Psi_{AA'BB'}\rangle = \iint \sum_{\lambda\sigma} g_{\lambda\sigma}(\mathbf{p}, \mathbf{q}) |\mathbf{p}\lambda\rangle_{AA'} |\mathbf{q}\sigma\rangle_{BB'} \tilde{\mathbf{d}}\mathbf{p} \tilde{\mathbf{d}}\mathbf{q}, \quad (11)$$

where $\tilde{\mathbf{d}}\mathbf{p}$ and $\tilde{\mathbf{d}}\mathbf{q}$ are the Lorentz-invariant momentum integration measures given by

$$\tilde{\mathbf{d}}\mathbf{p} \equiv \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2 + m^2}}, \quad (12)$$

and the functions $g_{\lambda\sigma}(\mathbf{p}, \mathbf{q})$ must satisfy

$$\sum_{\lambda\sigma} \iint |g_{\lambda\sigma}(\mathbf{p}, \mathbf{q})|^2 \tilde{\mathbf{d}}\mathbf{p} \tilde{\mathbf{d}}\mathbf{q} = 1. \quad (13)$$

To an observer in a frame Lorentz transformed by Λ^{-1} , the state $|\Psi_{AA'BB'}\rangle$ appears to be transformed by $\Lambda \otimes \Lambda$. Using Eq. (10), and a change of variables for \mathbf{p} , \mathbf{q} , λ , and σ , $g_{\lambda\sigma}(\mathbf{p}, \mathbf{q})$ goes through the following transformation:

$$g_{\lambda\sigma}(\mathbf{p}, \mathbf{q}) \rightarrow \sum_{\lambda'\sigma'} U_{\lambda,\lambda'}^{(\Lambda^{-1}\mathbf{p})} U_{\sigma,\sigma'}^{(\Lambda^{-1}\mathbf{q})} g_{\lambda'\sigma'}(\Lambda^{-1}\mathbf{p}, \Lambda^{-1}\mathbf{q}), \quad (14)$$

where we defined

$$U_{\lambda,\lambda'}^{(\mathbf{p})} \equiv D_{\lambda,\lambda'}^{(1/2)}(R(\Lambda, \mathbf{p})) \quad (15)$$

for compactness of notation. The Lorentz transformation can be viewed as a unitary operation, $R(\Lambda, \mathbf{p})$, conditioned on \mathbf{p} acting on the spin, followed by a boost $\mathbf{p} \rightarrow$

$\Lambda\mathbf{p}$ on the momentum represented by the circuit diagram in Fig. 1.

By writing $|\Psi_{AA'BB'}\rangle$ as a density matrix and tracing over the momentum degrees of freedom, the entanglement between A and B (that is, between the spin degrees of freedom) can be obtained by calculating Wootters's concurrence [18,19]

$$C(\rho_{AB}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (16)$$

where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are the square roots of the eigenvalues of the matrix $\rho_{AB}\tilde{\rho}_{AB}$, and $\tilde{\rho}_{AB}$ is the "time-reversed" matrix [18]

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y) \rho_{AB}^* (\sigma_y \otimes \sigma_y). \quad (17)$$

The first step in calculating the Lorentz-transformed concurrence is to find an explicit form for $U_{\lambda,\lambda'}^{(\mathbf{p})}$. Since any Lorentz transformation Λ can be written as a rotation $R(\Theta)$ followed by a boost $L(\xi)$ [see also Eq. (8)], it is clear that, for a pure rotation, $U_{\lambda,\lambda'}^{(\mathbf{p})}$ does not depend on \mathbf{p} . Hence, tracing over the momentum after a rotation will not change the concurrence. Therefore, we can look only at pure boosts, and without loss of generality we may choose boosts in the z direction. Writing the momentum 4-vector in polar coordinates as

$$\mathbf{p} = [E, p \cos(\theta) \sin(\phi), p \sin(\theta) \sin(\phi), p \cos(\phi)], \quad (18)$$

we obtain

$$U^{(\mathbf{p})} = \begin{bmatrix} \alpha & \beta e^{-i\theta} \\ -\beta e^{i\theta} & \alpha \end{bmatrix}, \quad (19)$$

where

$$\alpha = \sqrt{\frac{E+m}{E'+m}} \left[\cosh\left(\frac{\xi}{2}\right) + \frac{p \cos(\phi)}{(E+m)} \sinh\left(\frac{\xi}{2}\right) \right], \quad (20)$$

$$\beta = \frac{p \sin(\phi)}{\sqrt{(E+m)(E'+m)}} \sinh\left(\frac{\xi}{2}\right), \quad (21)$$

and

$$E' = E \cosh(\xi) + p \cos(\phi) \sinh(\xi). \quad (22)$$

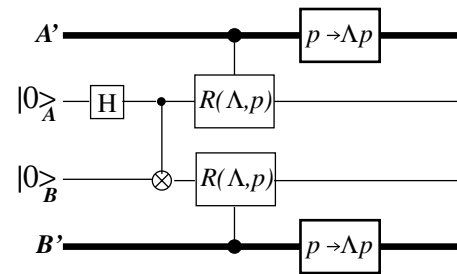


FIG. 1. Circuit diagram for a Lorentz boost on a state with spins in a $|\phi^+\rangle$ state. Lines representing momentum degrees of freedom are bold.

Here we use $\xi = |\boldsymbol{\xi}|$ as the rapidity of the boost in the z direction.

For momentum distributions in this Letter, we use a ‘‘relativistic Gaussian’’ with width σ_r ,

$$f(\mathbf{p}) = \sqrt{\frac{1}{N(\sigma_r)}} \exp\left(-\frac{p^2}{2\sigma_r^2}\right), \quad (23)$$

which differs from the standard Gaussian only in the normalization $N(\sigma_r)$, chosen in accordance with (12).

For a spin Bell state $|\phi^+\rangle$ with momenta in a product Gaussian, we have

$$g_{\lambda\sigma}(\mathbf{p}, \mathbf{q}) = \frac{1}{\sqrt{2}} \delta_{\lambda\sigma} f(\mathbf{p}) f(\mathbf{q}). \quad (24)$$

Boosting this state, we move some of the spin entanglement to the momentum. Tracing out the momentum from the Lorentz-transformed density matrix destroys some of the entanglement, and, hence, the concurrence in the moving frame diminishes. The change in concurrence depends only on the ratio σ_r/m and ξ . Figure 2 shows the concurrence vs rapidity ξ , for $\sigma_r/m = 1$ and 4. The decrease from the maximum value (the concurrence is one for Bell states) documents the boost-induced decoherence of the spin entanglement.

In the limit $\xi \rightarrow \infty$ (boost to the speed of light), the concurrence saturates, i.e., it reaches a constant value that depends on the mass of the particles and the shape of the momentum distribution. In this particular example, it depends on the ratio σ_r/m . The saturation level decreases as σ_r/m increases until $\sigma_r/m \simeq 3.377$ when the saturation level becomes zero. Note that in the limit of ‘‘pure’’ momentum states (plane waves), the spins undergo local unitary rotations but entanglement transfer does not occur, as was observed in [20,21]. The reason for the saturation can be seen by examining (19) in the limit $\xi \rightarrow \infty$,

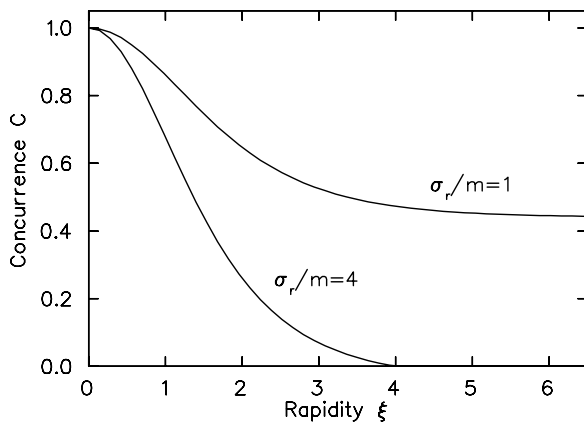


FIG. 2. Spin concurrence as a function of rapidity, for an initial Bell state with momenta in a product Gaussian. Data is shown for $\sigma_r/m = 1$ and $\sigma_r/m = 4$.

$$\lim_{\xi \rightarrow \infty} \alpha = \sqrt{\frac{E+m}{2[E+p \cos(\phi)]}} \left(1 + \frac{p \cos(\phi)}{E+m}\right), \quad (25)$$

$$\lim_{\xi \rightarrow \infty} \beta = \frac{p \sin(\phi)}{\sqrt{2(E+m)[E+p \cos(\phi)]}}. \quad (26)$$

The parameter β represents the amount of rotation due to the boost. If we maximize β with respect to ϕ , we obtain

$$\beta \rightarrow \frac{p/m}{1 + \sqrt{1 + (p/m)^2}}, \quad (27)$$

which is a monotonically increasing function of p/m . For a particle of mass m and magnitude of momentum p , Eq. (27) represents the maximal amount of rotation due to a boost. By increasing σ_r/m in our example, we effectively increase this limit for β , and, hence, how much we can alter the concurrence. Note that since for large p/m the rotation β tends to one, the Lorentz transformation is equivalent to a conditional spin flip in this regime.

If boosts can disentangle spins, can they transfer entanglement from the momentum degrees of freedom to an unentangled spin wave function? Indeed this is possible. One way to achieve this is to take any of the resulting states after boosting the state in Eq. (24) and apply the inverse boost to increase concurrence to one. Note that the increase in spin entanglement comes at the expense of a loss of momentum entanglement, since the entanglement between all degrees of freedom (spin and momentum) is constant under Lorentz transformations.

Simply reversing a previously applied Lorentz transformation as in the last example is not a very satisfying way to create entanglement. Is there a way we could create an unentangled state in the laboratory frame that would appear entangled to a moving observer? Consider the state

$$|\Psi_{AA'BB'}\rangle = \frac{1}{\sqrt{2}} (|\mathbf{p}, -\mathbf{p}\rangle |\phi^+\rangle + |\mathbf{p}_\perp, -\mathbf{p}_\perp\rangle |\phi^-\rangle), \quad (28)$$

where \mathbf{p} and \mathbf{p}_\perp are both in the x, y plane, have the same magnitude p , and are perpendicular. We could imagine such a state arising from a particle decay where the products are restricted to movement in the x or y axes with a conditional σ_z gate on the perpendicular direction. The reduced density matrix ρ_{AB} for this wave function is separable, and its concurrence vanishes. However, taking the large p limit in Eqs. (25) and (26) and choosing ϕ and θ appropriate for \mathbf{p} and \mathbf{p}_\perp in the x and y directions, respectively, one can show that for a large boost in the z direction both $|\phi^+\rangle$ and $|\phi^-\rangle$ are transformed into the $|\psi^-\rangle$ state and, hence, the spins are maximally entangled in this reference frame. In fact, the concurrence as a function of \mathbf{p} and ξ is given by

$$C(\rho_{AB}) = \frac{p^2 [\cosh^2(\xi) - 1]}{[\sqrt{1 + p^2 \cosh(\xi)} + 1]^2}, \quad (29)$$

when choosing $m = 1$. Note that the concurrence is

greater than zero whenever p and ξ are nonzero, and as p and ξ become large the concurrence tends to one. So, if we restrict ourselves to spin measurements, an observer in the rest frame of the decay particle cannot use entanglement as a resource (e.g., for teleportation, superdense coding, etc.) while the moving observer *can*. Such a purification of spin entanglement is not always possible, however, and the following theorem characterizes the limitations.

Theorem.—The entanglement *between* the spin and momentum parts of a pure state wave function, $|\Psi_{AA'BB'}\rangle$, must be nonzero to allow the spin entanglement to increase under Lorentz transformations.

Proving the contrapositive, starting with a product state of the form

$$|\Psi_{AA'BB'}\rangle = |\psi\rangle_{A'B'}|\phi\rangle_{AB}, \quad (30)$$

and applying boosts of the form $\Lambda \otimes \Lambda$ or even $\Lambda \otimes \Lambda'$, we obtain

$$\rho_{AB} = \sum_i p_i U_A^i \otimes V_B^i |\phi\rangle\langle\phi| U_A^{i\dagger} \otimes V_B^{i\dagger}, \quad (31)$$

where U_A^i and V_B^i are unitary operators and the sum $\sum_i p_i$ will be an integral for certain states $|\psi\rangle$. We can now plug Eq. (31) into any entanglement monotone $E(\rho)$ and obtain the inequality,

$$E(\rho_{AB}) \leq \sum_i p_i E(U_A^i \otimes V_B^i |\phi\rangle) \quad (32)$$

$$= \sum_i p_i E(|\phi\rangle) \quad (33)$$

$$= E(|\phi\rangle), \quad (34)$$

where inequality (32) comes from the definition of an entanglement monotone [22]. Hence, for states of the form Eq. (30), the spin entanglement can only *decrease* after a Lorentz transformation. \square

Note that this theorem does not hold if arbitrary unitary operations are applied to a particle's spin and momentum degrees of freedom (for instance, a swap gate), but it does hold for the entire class of unitaries realized by Lorentz transformations.

We have investigated the properties of moving entangled pairs of massive particles. Because Lorentz boosts entangle the spin and momentum degrees of freedom, entanglement can be transferred between them. This is true for single particles [14], and we have shown here that it is true for pairs, where the Lorentz boost affects the entanglement *between* spins. Quite generally, we can say that fully entangled spin states will (depending on the initial momentum wave function) most likely decohere due to the mixing with momentum degrees of freedom. We also note, however, that such mixing can *purify* spin entanglement if the momentum degrees are entangled with the spin. The physics of creating entanglement be-

tween spins and between momenta is very different. Thus, the possibility of entanglement transfer via Lorentz boosts could conceivably, in special situations, lead to simplified state preparation and purification protocols.

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- [1] A. Einstein, *Jahrb. Radioakt. Elektron.* **4**, 411 (1907); M. Planck, *Ann. Phys. (Leipzig)* **26**, 1 (1908).
 - [2] H. Ott, *Z. Phys.* **175**, 70 (1963); H. Arzeliès, *Nuovo Cimento* **35**, 792 (1964).
 - [3] R. Aldrovandi and J. Gariel, *Phys. Lett. A* **170**, 5 (1992).
 - [4] K. P. Trout and A. J. Greiner, *Nuovo Cimento* **113**, 1439 (1998).
 - [5] P. T. Landsberg and G. E. A. Matsas, *Phys. Lett. A* **223**, 401 (1996).
 - [6] P. J. B. Peebles and D. T. Wilkinson, *Phys. Rev.* **174**, 2168 (1968).
 - [7] C. E. Shannon, *Bell Syst. Tech. J.* **27**, 379 (1948); **27**, 623 (1948).
 - [8] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, England, 2000).
 - [9] C. H. Bennett *et al.*, *Phys. Rev. Lett.* **70**, 1895 (1993).
 - [10] C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).
 - [11] R. Josza, D. S. Abrams, J. P. Dowling, and C. P. Williams, *Phys. Rev. Lett.* **85**, 2010 (2000); V. Giovannetti, S. Lloyd, and L. Maccone, *Nature (London)* **412**, 417 (2001).
 - [12] B. Julsgaard, A. Kozhekin, and E. S. Polzik, *Nature (London)* **413**, 400 (2001).
 - [13] N. J. Cerf and C. Adami, *Phys. Rev. Lett.* **79**, 5194 (1997).
 - [14] A. Peres, P. F. Scudo, and D. R. Terno, *Phys. Rev. Lett.* **88**, 230402 (2002).
 - [15] W. Tung, *Group Theory in Physics* (World Scientific, New York, 1985).
 - [16] G. Vidal, *J. Mod. Opt.* **47**, 355 (2000).
 - [17] S. Parker, S. Bose, and M. B. Plenio, *Phys. Rev. A* **61**, 032305 (2000).
 - [18] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
 - [19] After the Lorentz transform the spins are in a mixed state; entanglement measures based on pure states cannot be used.
 - [20] M. Czachor, *Phys. Rev. A* **55**, 72 (1997).
 - [21] P. M. Alsing and G. J. Milburn, *Quantum Inf. Comput.* **2**, 487 (2002).
 - [22] G. Vidal, *J. Mod. Opt.* **47**, 355 (2000).