

Quantum Computation with Linear Optics

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Abstract. We present a constructive method to translate small quantum circuits into their optical analogues, using linear components of present-day quantum optics technology only. These optical circuits perform precisely the computation that the quantum circuits are designed for, and can thus be used to test the performance of quantum algorithms. The method relies on the representation of several quantum bits by a single photon, and on the implementation of universal quantum gates using simple optical components (beam splitters, phase shifters, etc.). The optical implementation of Brassard et al.'s teleportation circuit, a non-trivial 3-bit quantum computation, is presented as an illustration.

1 Introduction

The promise of ultrafast computation using quantum mechanical logic raised by Shor's discovery of a polynomial algorithm for factoring [1] has yet to materialize in physical implementations. While quantum logic has been implemented in a number of different guises [2,3], the dynamics and behavior of a quantum circuit subject to noise and quantum decoherence has only been tested in simulations on a classical computer [4,5] (but see [6]). There is little controversy about the realization that it is the quantum mechanical superposition principle, and the entangled, nonlocal, states it engenders, that are at the origin of the speed-up of quantum algorithms with respect to their classical counterparts. Still, effective quantum algorithms are few and far between, and even those that are known today have yet to be tested in a physical realization (but see [7].)

In anticipation of physical realizations that implement quantum superpositions between physical states, we present here a method of constructing circuits based on non-local superpositions of "eventualities", rather than physical objects. More precisely, we *simulate* quantum superpositions, "qubits", as "which-path" eventualities in linear optics, implementable on standard optical benches. While the "support" of these qubits is decidedly classical (the optical devices such as beam splitters, polarizers, etc.) the wave function at the exit of the optical circuit can be made to coincide arbitrarily well with the outcome of the anticipated computation, thus implementing the quantum circuit. Naturally, this "classical" implementation of quantum logic has its drawbacks, as we comment on further below. Still, it should provide an excellent (and cost effective) means for testing small circuits for quantum error correction or quantum algorithms.

As we point out below, the realization that optical which-path eventualities of single photons can simulate qubits is not new in itself. Here, we focus on *protocols* to translate *any* quantum circuit diagram into *linear* optics networks, which puts the realization of simple circuits decidedly within reach. Quantum computation can be described as the task of performing a specific unitary transformation on a set of quantum bits (qubits) followed by measurement, so that the outcome of the measurement provides the result of the computation. This unitary transformation can be constructed with a finite number of 4×4 unitary matrices, that is, using a quantum circuit utilizing only 1-bit and 2-bit quantum gates (see, *e.g.*, [8,9]). The universality of 1- and 2-bit gates in the realization of an arbitrary quantum computation was shown in [10]. Furthermore, it was realized recently that an *optical* realization exists for any $N \times N$ unitary matrix [11], a result which generalizes the well-known implementation of $U(2)$ matrices using a lossless beam splitter and a phase shifter (see, *e.g.*, [12]). Accordingly, each element of $U(N)$ can be constructed using an array of $\mathcal{O}(N^2)$ beam splitters that form an optical multipoint with N input and N output beams. As we shall see below, this result together with the universality of (1- and 2-bit) gates, can be exploited *constructively*, providing a systematic method for assembling optically-simulated gates to build simple quantum circuits.

2 Logical Qubits in Optics

Let us start by considering the equivalence between traditional linear optics elements (such as beam splitters or phase shifters) and 1-bit quantum gates (see, *e.g.*, [13]). This equivalence is inspired by the standard two-slit experiment of quantum mechanics, in which a single quantum can interfere with itself to produce fringes on a screen. Accordingly, a quantum on the other side of the slit is in a superposition of paths, and the quantum mechanical uncertainty principle is in full effect with respect to location and phase [14].

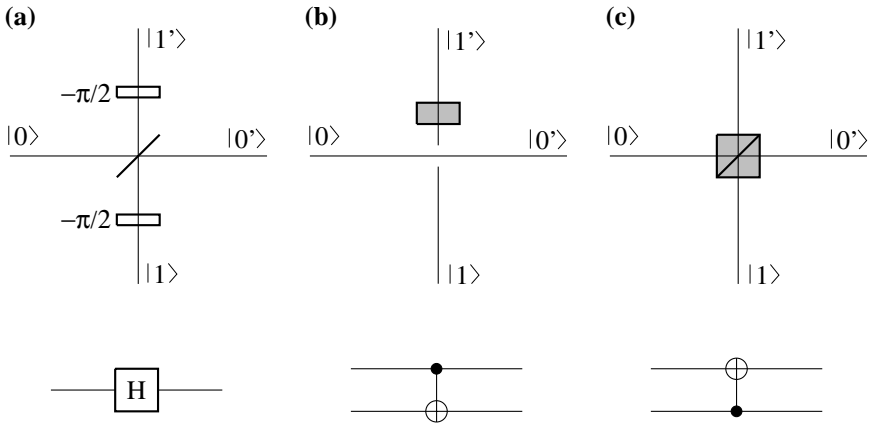
For example, in quantum circuit terminology, an optical symmetric beam splitter is known to act as a quantum $\sqrt{\text{NOT}}$ gate (up to a phase of $\pi/4$) if we use the pair of input modes $|01\rangle$ (or $|10\rangle$) to represent the logical 0 (or 1) state of the qubit. If one input port is in the vacuum state $|0\rangle$ (absence of a photon) and the second one is in a single-photon state $|1\rangle$, the output ports will then be in a superposition state $|01\rangle + i|10\rangle$. Thus, with the identification of the *logical* qubits $|0_L\rangle \equiv |01\rangle$ and $|1_L\rangle \equiv |10\rangle$, we produce the wavefunction

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0_L\rangle + i|1_L\rangle) , \quad (1)$$

from the initial state $|0_L\rangle$ just by running a photon through a beam splitter. (The factor i arises from the $\pi/2$ phase shift between the transmitted and the reflected wave in a lossless symmetric beam splitter [?].)

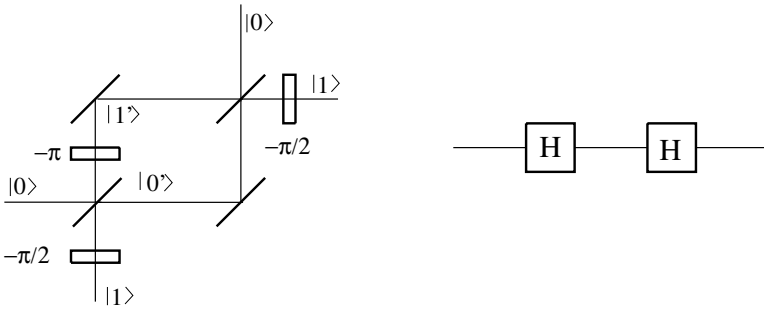
Similarly, a quantum phase gate can be obtained by use of a phase shifter acting on one mode of the photon. In other words, single-photon interferometry experiments can be interpreted in quantum circuit language, the “which-path”

Fig. 1. Example of optical simulation of basic quantum logic gates. (a) Hadamard gate on a “location” qubit, using a lossless symmetric beam splitter. (b) Controlled-NOT gate using a polarization rotator. The location and polarization are the control and target qubit, respectively. (c) Same as (b) but the control and target qubits are interchanged by the use of a polarizing beam splitter.



variable being substituted with a quantum bit. Although a general proof for the existence of an optical realization of an arbitrary quantum circuit is implicitly given in Ref. [11], the simple duality between quantum logic and single-photon optical experiments has not been exploited. Here (and in [16]) it is shown that a *single-photon* representation of *several* qubits can be used to exploit this duality: as several (say n) quantum bits can be represented by a single photon in an interferometric setup involving essentially 2^n paths, quantum conditional dynamics can easily be implemented by using different optical elements in distinct paths. The appropriate cascading of beam splitters and other linear optical devices entails the possibility of simulating networks of 1- and 2-bit quantum gates (such as the Hadamard or the controlled-NOT gate, see Fig. 1), and thereby in principle achieving universal n -bit quantum computations [16]. This is in contrast with traditional optical models of quantum logic, where in general n photons interacting through *nonlinear* devices (acting as 2-bit quantum gates) are required to represent n qubits (see, *e.g.*, [13]). Such models typically make use of the Kerr nonlinearity to produce intensity-dependent phase shifts, so that the presence of a photon in one path induces a phase shift to a second photon (see, *e.g.*, the optical realization of a Fredkin gate [17]). Instead, the method proposed here yields a straightforward method for “translating” *any* n -bit quantum circuit into a single-photon optical setup, whenever n is not too large. The price to pay is the exponential growth of the number of optical paths, and, consequently, of

Fig. 2. Implementation of two sequential Hadamard transformations as a balanced Mach-Zehnder interferometer using lossless symmetric beam splitters only.



optical devices that are required. This will most likely limit the applicability of the proposed technique to the implementation of relatively simple circuits.

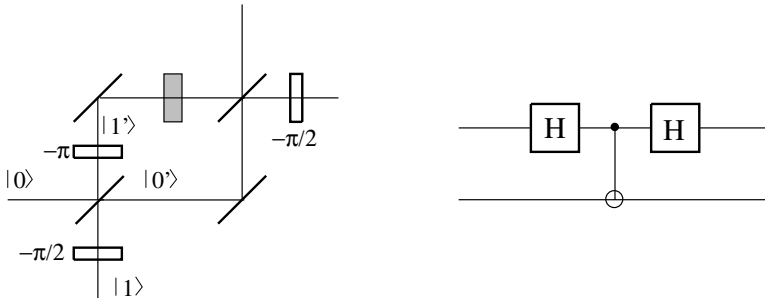
First, let us consider a single-photon experiment with a Mach-Zehnder interferometer in order to illustrate the optical simulation of elementary quantum gates (see Fig. 1). One qubit is involved in the description of the interferometer in terms of a quantum circuit: the “location” qubit, characterizing the information about “which path” is taken by the photon. Rather than using the occupation number representation for the photon, here we label the two input modes entering the beam splitter by $|0\rangle$ and $|1\rangle$ (“mode description” representation). The quantum state of the photon *exiting* the beam splitter then is $|0'\rangle + i|1'\rangle$ or $|1'\rangle + i|0'\rangle$ depending on the input mode of the photon. This is the $\sqrt{\text{NOT}}$ gate discussed earlier. Placing phase shifters at the input and output ports as shown in Fig. 1a, the beam splitter can be shown to perform a Hadamard transformation between input and output modes, *i.e.*,

$$\begin{pmatrix} |0'\rangle \\ |1'\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \tag{2}$$

In this sense, a lossless symmetric beam splitter (supplemented with two $-\pi/2$ phase shifters) can be viewed as a Hadamard gate acting on a location qubit. Recombining the two beams using a second beam splitter (see Fig. 2) in order to form a balanced Mach-Zehnder interferometer corresponds therefore, in this quantum circuit language, to having a second Hadamard gate acting subsequently on the qubit¹. Since $H^2 = 1$, it is not a surprise that the location qubit returns to the initial basis state ($|0\rangle$ or $|1\rangle$) after two beam splitters (with the appropriate phase shifter). This sequence of two Hadamard gates simply conveys the fact that the contributions of the two paths interfere destructively in

¹ Here and below, it is understood that the path lengths are adjusted so that the difference between dynamical phases vanishes.

Fig. 3. Implementation of two sequential Hadamard transformations with intermediate *conditional* operation on the polarization. This circuit produces an entangled state $|0\rangle_{\text{pol}}|0\rangle_{\text{loc}} + |1\rangle_{\text{pol}}|1\rangle_{\text{loc}}$ between the polarization (which is initially in a product state with the location qubit at the input port of the interferometer) and the location qubit (denoted by $|0\rangle$ and $|1\rangle$ in the figure) just before the final beam splitter, preventing the observation of interference fringes at the output of this interferometer.



one of the output ports, so that the photon always leaves the interferometer in the other.

More interestingly, consider now the same interferometer using polarized photons (the photon is horizontally polarized at the input). Assuming that none of the devices acts on polarization, the photon exits the interferometer with the same polarization. In a circuit terminology, this corresponds to introducing a “polarization” qubit ($|0\rangle_{\text{pol}}$ stands for horizontal polarization) which remains in a product state with the location qubit throughout the circuit. If a polarization rotator is placed in one of the branches of the interferometer, flipping the polarization from horizontal $|0\rangle_{\text{pol}}$ to vertical $|1\rangle_{\text{pol}}$, it is well known that interference disappears since both paths become distinguishable. This corresponds to placing a 2-bit controlled-NOT gate (represented in Fig. 1b) between the two Hadamard gates, where the location qubit is the control and polarization is the target bit (see Fig. 3). The circuit in Fig. 3 thus simply implements the dynamics

$$|0\rangle_{\text{pol}}|0\rangle_{\text{loc}} \rightarrow \frac{1}{\sqrt{2}} (|0\rangle_{\text{pol}}|0\rangle_{\text{loc}} + |1\rangle_{\text{pol}}|1\rangle_{\text{loc}}) \tag{3a}$$

$$\rightarrow \frac{1}{2} (|0\rangle_{\text{pol}}|0\rangle_{\text{loc}} + |1\rangle_{\text{pol}}|0\rangle_{\text{loc}} + |0\rangle_{\text{pol}}|1\rangle_{\text{loc}} - |1\rangle_{\text{pol}}|1\rangle_{\text{loc}}) . \tag{3b}$$

which “tags” each path with a particular polarization just before the final beam splitter in the sense that the polarization of the photon is flipped *conditionally* on its location. The disappearance of interference fringes then simply reflects the entanglement between location and polarization qubits (the reduced density matrix obtained by tracing over polarization shows that the photon ends up in a mixed “location” state, *i.e.*, it has a 50% chance of being detected in one or

the other exit port). This suggests that Feynman's rule of thumb (namely that interference and which-path information are complementary) is a manifestation of the quantum *no-cloning* theorem: the location qubit cannot be "cloned" into a polarization qubit. However, the fringes can be resurrected via a *quantum erasure* procedure [18] (which involves placing polarizing beam-splitters, introduced below, at the exit ports of the construction).

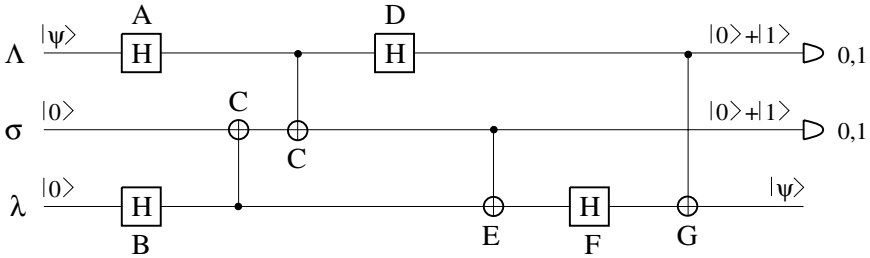
The optical analogue of other basic quantum gates can be devised following the same lines. For example, a polarizing beam splitter achieves a controlled-NOT gate where the location qubit is flipped or not (the photon is reflected or not) conditionally on its state of polarization, as shown in Fig. 1c. Fredkin, Toffoli, as well as controlled-phase gates can easily be simulated in the same manner but will not be considered here. The central point is that, in principle, a universal quantum computation can be simulated using these optical substitutes for the universal quantum gates. The optical setup is constructed straightforwardly by inspection of the quantum circuit. A circuit involving n qubits requires in general n successive splitting stages of the incoming beam, that is, 2^n optical paths are obtained via $2^n - 1$ beam splitters. (The use of polarization of the photon as a qubit allows using 2^{n-1} paths only.) This technique is thus limited to the simulation of quantum networks involving a relatively small number of qubits (say less than 5-6 with present technology). The key idea of a quantum computer, however, is to avoid just such an exponential size of the apparatus by having n physical qubits performing unitary transformations in a 2^n -dimensional space. In this respect, it can be argued that an optical setup requiring $\sim 2^n$ optical elements to perform an n -bit quantum computation represents a *classical* optical computer (see, *e.g.*, [9]). Accordingly, the issue of whether non-locality (which is at the heart of entanglement) is *physically* present in the optical realization is a matter of debate.

3 Optical Quantum Teleportation

As an illustration, we show that a quantum circuit involving 3 qubits and 8 quantum gates (see Fig. 4) can be implemented optically using essentially 9 beam splitters [16]. This circuit² has the property that the arbitrary initial state $|\psi\rangle$ of qubit A is "teleported" to the state in which qubit λ is left after the process. In the original teleportation scheme [20], two classical bits (resulting from a Bell measurement) are sent by the emitter, while the receiver performs a specific unitary operation on λ depending on these two bits. However, it is shown in [21] that these unitary operations can be performed at the quantum level as well, by using quantum logic gates and postponing the measurement of the two bits to the end of the circuit. The resulting quantum circuit (Fig. 4) is *formally* equivalent to the original teleportation scheme (although no classical bits are communicated) as exactly the same unitary transformations and quantum gates are involved. While we do not claim that an optical realization gives rise to "genuine" teleportation, this example circuit is instructive to demonstrate

² This teleportation circuit is equivalent to the one described in [19].

Fig. 4. Quantum circuit for teleportation (from [19]). The initial state of qubit A is teleported to the state of qubit λ . Qubits σ and λ must be initially in state $|0\rangle$. Qubits A and σ , if measured at the end of the circuit, yield two classical (random) bits that are uniformly distributed.



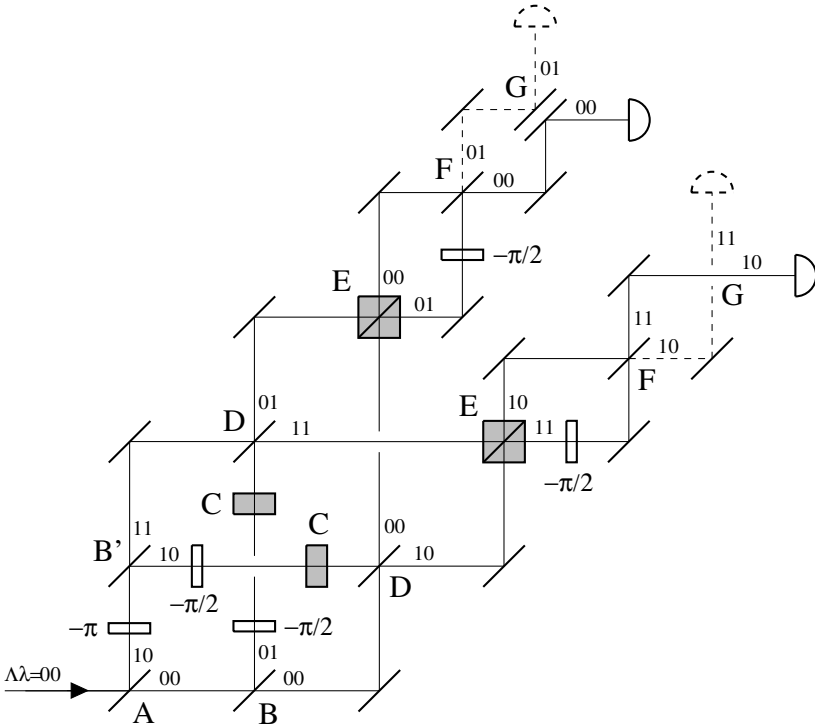
the correspondence between quantum logic and optical devices as it is small (3 qubits) but non-trivial.

In the optical counterpart of this circuit (see Fig. 5), qubits A and λ correspond to the location of the photon at the first and second splitting level, while σ stands for the polarization qubit. Note that the photons are initially horizontally polarized, *i.e.*, in polarization state $|0\rangle$. The first beam splitter A in Fig. 5 acts as a Hadamard gate on A , as explained previously. For convenience, we depict the teleportation of state $|\psi\rangle = |0\rangle$, so that the incident photon enters this beam splitter in the input port labeled $|0\rangle$. However, as any operation in $U(2)$ can be realized optically, an *arbitrary* state of A can be prepared (and then teleported) by having an additional beam splitter (with tunable phase shifters) connected to both input ports of beam splitter A . The second level of beam splitters B (and B')³ corresponds to the Hadamard gate B on λ in Fig. 4. The four paths at this point ($A\lambda = 00, 01, 10,$ and 11) label the four components of the state vector characterizing qubits A and λ . The probability amplitude for observing the photon in each of these four paths, given the fact the photon enters the $|0\rangle$ port of beam splitters A and B , is then simply the corresponding component of the wave vector. The combined action of both controlled-NOT gates C in Fig. 4 is to flip the polarization state of the photon (qubit σ) conditionally on the parity of $A + \lambda \pmod 2$, which is achieved by inserting polarization rotators C at the appropriate positions. In other words, the polarization is flipped on path 01 or 10 , while it is unchanged on path 00 or 11 .

The Hadamard gate D in Fig. 4 acts on qubit A , independently of λ . This is achieved in Fig. 5 by grouping the paths in pairs with the same value of λ (*i.e.*, crossing the paths) and using two beam splitters D in order to effect a Hadamard transformation on A (one for each value of λ). Similarly, the controlled-NOT gate E acting on λ (conditionally on the polarization) is implemented by the use

³ For convenience, two realizations (B and B') of the Hadamard gate are used in Fig. 5, where B' is obtained from B by interchanging the $|0'\rangle$ and $|1'\rangle$ output ports in Fig. 1a.

Fig. 5. Optical realization of the quantum circuit for teleportation using polarized photons. The location qubit λ characterizes the “which-arm” information at the first beam splitter, while qubit λ stands for the “which-path” information at the second level of splitting. The initial location qubit λ is teleported to qubit λ and probed via the interference pattern observed at the upper or lower ($\lambda = 0, 1$) final beam splitter, for both polarization states ($\sigma = 0, 1$) of the detected photon.



of two polarizing beam splitters E after crossing the paths again. A polarizing beam splitter leaves a horizontally polarized photon (state $|0\rangle$) unchanged, while vertical polarization (state $|1\rangle$) is reflected. The last Hadamard gate F in Fig. 4 corresponds to the last two beam splitters F, and the final controlled-NOT gate G is simply achieved by crossing the paths ($\lambda = 0, 1$) in the lower arm ($\lambda = 1$) versus the upper arm ($\lambda = 0$). In fact, the setup could be simplified by noting that the conditional crossing of paths achieved by G simply reduces to relabeling the output ports of beam splitter F in the $\lambda = 1$ arm. In Fig. 5, only those phase shifters associated with the Hadamard gates (Fig. 1a) that are relevant in the final detection are indicated.

The interpretation of this optical circuit in the language of teleportation is the following. After being “processed” in this quantum circuit, a photon which was initially horizontally polarized can reach one of the two “light” detectors

(solid line in Fig. 5) with horizontal or vertical polarization. This corresponds to the final measurement of qubits A and σ in Fig. 4, yielding two classical (random) bits: upper or lower arm, horizontal or vertical polarization. The third qubit, λ , contains the teleported quantum bit, that is, the initial arbitrary state of A . Since the location state of the photon is initially $|0\rangle$ in the setup represented in Fig. 5, it always exits to the “light” detector and never reaches the “dark” one (dashed line). For any measured value of A (photon detected in the upper or lower arm) and σ (horizontally or vertically polarized photon), the entire setup forms a simple balanced Mach-Zehnder interferometer. Indeed, there are exactly two *indistinguishable* paths leading to each of the eight possible outcomes (four detectors, two polarizations); these interfere pairwise, just as in a standard Mach-Zehnder interferometer, explaining the fact that the photon always reaches the “light” detector (in both $A = 0$ and $A = 1$ arms and for both polarizations). In this sense, the initial “which-arm” qubit A has been teleported to the final “which path” qubit λ . Note that, as no photodetection coincidence is required in this optical experiment, the setup is actually not limited to *single*-photon interferometry. This largely simplifies the realization of the optical source since classical light fields (such as those from a laser) can be used rather than number states.

4 Conclusion

We have proposed a general technique for simulating small-scale quantum networks using optical setups composed of linear optical elements only. This avoids the recourse to non-linear Kerr media to effect quantum conditional dynamics, a severe constraint in the usual optical realization of quantum circuits. A drawback of this technique is clearly the exponential increase of the resources (optical devices) with the size of the circuit. Nevertheless, as optical components that simulate 1- and 2-bit universal quantum gates can be cascaded straightforwardly, a non-trivial quantum computing optical device can easily be constructed if the number of component qubits is not too large. We believe this technique can be applied without fundamental difficulties to the encoding and decoding circuits that are involved in the simplest quantum error-correcting schemes [22], opening up the possibility for an experimental simulation of them. Furthermore, this technique promises a technologically simple way to test quantum algorithms for performance and error stability. Last but not least, the correspondence between quantum circuits and optical (interferometric) setups suggests that new and improved interferometers could be designed using the quantum circuit language [23].

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