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Complexity, Computation, and Measurement

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We consider the duality between computation and measurement emerging from the thermodynamics of Turing machines. This duality suggests a new measure of complexity (physical complexity) which corresponds to our intuition and appears to be practical enough to estimate the complexity of genomes. From an automata theoretic point of view, this complexity is just the mutual information between a thermodynamic Turing machine and the equilibrated tape that constitutes its "universe". By the same token, the complexity is dual to the mutual entropy between the observer and the system in the information theory of measurement. The complexity of a bit-string, therefore, is the number of bits that can be used to lower the entropy of a closed system. Random strings have vanishing physical complexity (even though they have maximal Kolmogorov complexity), as they cannot be the result of a (classical) measurement or computation.

1 Introduction

The study of "complex systems", or more generally the science of "complexity", has enjoyed tremendous growth in the last ten years, despite the fact that complexity itself is only vaguely defined, and many alternatives have surfaced over the years[1]. For this reason alone, no real progress could be achieved in the understanding of the emergence, and growth, of complexity in the living world, for example. While this complexity, and the apparent order it entails, is ubiquitous, physicists have struggled for many decades to understand how it could be compatible with the second law of thermodynamics [16]. Much progress has been achieved since then, driven by the pioneering work of Landauer [12] and Bennett [4], culminating in the resolution of the Maxwell demon paradox. In order to guarantee progress in the study of complexity, we must insist that the term is rigorously defined, certainly as rigorously as entropy or information. One of the reasons for the persistent misunderstanding generated by the inaccurate use of these terms, we believe, lies in the fact that Shannon's information theory [17] is usually viewed by physicists to belong to the realm of computer science and engineering. One of the points we would like to make here is to insist that information theory is part of statistical physics, more specifically part of thermodynamics [13]. As has been verbalized by many before us, it is *not* an accident that Shannon entropy appears so similar to Boltzmann-Gibbs entropy. They are one and the same, and we will refer to it henceforth as Boltzmann-Gibbs-Shannon (BGS) entropy. Shannon had the perspicacity to introduce the notions of *conditional* and *mutual* entropies. Along with this, Shannon realized that mutual entropy allows for an unambiguous definition of information: information *is* mutual entropy. Besides information science, these constructions have immediate relevance for thermodynamics as well, but have been largely ignored by physicists. They allow for the consistent thermodynamic description of the interacting *parts* of a closed system. As we shall recapitulate below, the "rediscovery" of conditional and mutual entropy in thermodynamics has led to the resolution of the Maxwell demon paradox. In the light of the arguments presented below, and in order to stave off future misunderstandings, we feel compelled to reiterate this point: information has a precise definition in physics, and it is given by the mutual entropy between two systems. Information is *not* a list of symbols. In other words, without referring to an environment that a string is to be interpreted within, the notion of information is meaningless. It is this simple fact that will permit us to present a consistent, intuitive, and possibly practical measure of complexity.

In the next section we discuss algorithmic (Kolmogorov) complexity to fix notation and point out its shortcomings as a measure of physical complexity. In the third section we recapitulate the physics of measurement and the resolution of the Maxwell demon paradox, in order to set the stage for the exposition of the duality between computation and measurement (a notion anticipated 35 years earlier by Landauer[2]). We then introduce physical complexity in Section 4, and use it to estimate the complexity of a well-known piece of RNA in Section 5. We finish with some comments and conclusions.

2 Kolmogorov Complexity

Kolmogorov-Chaitin (KC) complexity is rooted in automata theory [19], and provides a measure for the *regularity* of an alphabet string. Briefly, a string is said to

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[1]See, e.g., [11, 7, 5, 14, 8]

[2]In his pioneering paper [12], Landauer writes: "The computing process, where the setting of various elements depends upon the setting of other elements at previous times, is closely akin to a measurement".

be "regular" if the algorithm necessary to produce it on a universal (Turing) automaton is shorter (with length measured in bits) than the string itself. A simple example is a bit string with a repetitive pattern, such as 1010101010... The minimal "program" enabling a Turing automaton to write this string only requires the pattern 10 and "repeat, write" instructions. A less obvious example is the binary equivalent of the first one hundred (say) digits of π. While random *prima facie*, a succinct algorithm for a Turing machine can be written; as a consequence such a string is classified as regular. Technically, the KC-complexity of a string s is defined (in the limit of long strings) as the length (in bits) of the shortest program p producing s when run on universal Turing machine T:

$$K(s) = \min\left(|p|:\ s = C_T(p)\right) \tag{1}$$

where $|p|$ denotes the length of the program in bits. While KC-complexity is only defined modulo the number of instructions (on a Turing machine) necessary to simulate any other computer, it becomes exact in the limit of infinite strings[3].

A simple consequence of (1) is that algorithmically regular strings have vanishing KC-complexity in the limit of infinite strings, while "random" strings (such as binary strings obtained from a coin-flip procedure) are assigned *maximum* KC-complexity, i.e., for a random string r:

$$K(r) \approx |r|\ . \tag{2}$$

Physically, and intuitively, this is unsatisfactory. What is really in question here is the very definition of "random" in a *physical* world. Our intuition demands that the complexity of a random string ought to be zero, as it is somehow "empty". It is generally believed that the impossibility to produce random numbers is one of the hallmarks of (classical) computers, in contrast to (2), where a random string is purportedly *computed* by a Turing machine. From an automata-theoretic point of view then, Eq. (2) is meaningless as random numbers are *uncomputable*. At the very least, the complexity of a random string should be undecidable.

It is thus inevitable to conclude that *algorithmic randomness*, the measure of KC-complexity, while well-defined for non-random strings, is irrelevant for physical systems. In its place, we propose a measure that can be defined as rigorously as KC-complexity using automata theory, but also conforms to the thermodynamics of measurement.

[3] A program executable on Turing machine T can also be executed (with the same result) on any other universal computer T', provided that it is preceded by a *prefix* code. The difference in size of the minimal program on T and T' due to the length of the prefix can be string dependent, but vanishes in the limit of infinite strings.

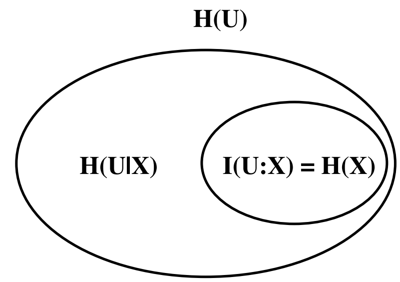

Figure 1: Entropy diagram for the measurement situation.

3 Computation and Measurement

Physically, we can define a random number (or, more generally, a string) as one that cannot be obtained by a (classical) *measurement* performed on a physical system. Using the thermodynamics of measurement, this statement can be made very precise. Consider a closed system U (the "universe", for short)[4], and the uncertainty—or BGS entropy—$H(U)$ associated with it. Performing a measurement on U entails correlating part of U's degrees of freedom with a measurement device which carries a pointer with degrees of freedom X. After recording the state of X, the uncertainty of the system excluding X is $H(U|X)$, i.e., the *conditional* entropy of U *knowing* X. The information conveyed about U by measuring X is $I(U:X)$, i.e., the mutual entropy between U and X. Because in this case X is entirely comprised in U, $I(U:X)$ happens to be equal to the entropy of X (see Fig. 1). As is apparent from Fig. 1, the entropies are related by the fundamental equation of measurement

$$I(U:X) = H(U) - H(U|X)\ . \tag{3}$$

As was shown by Landauer [12] and emphasized by Bennett [4], the measurement process does not violate the second law of thermodynamics, as the decrease in entropy $\Delta S = H(U) - H(U|X)$ is precisely compensated by the heat dissipated when recording the information $I(U:X)$. This is illustrated succinctly with Szilard's *gedankenexperiment* [18], designed to refute the possibility of a Maxwell-demon. Imagine a single molecule enclosed in a box which is in thermal contact with a heat bath (Fig. 2). A piston can be inserted into the box separating it into two volumes. After determining on which side of the piston the molecule is located (the measurement), the piston is moved in a direction determined by the outcome of the experiment, namely in such a way that the kinetic pressure of the molecule assists the movement of the piston. As this operation proceeds isothermally, the reduction of entropy (achieved through knowledge of the

[4] We call a closed system "universe", as we think of it as the environment in which the string is to be interpreted. This means that in general there may be systems *outside* of U, even though this is not possible for the physical universe.

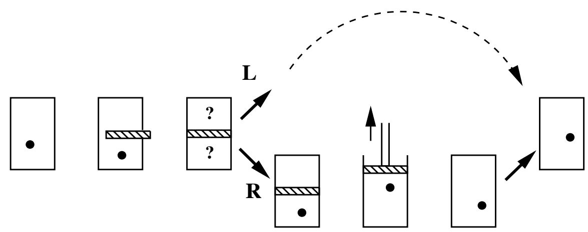

Figure 2: Szilard's one-molecule engine, designed to extract $T\log 2$ of work per cycle via measurement of the position of the molecule (adapted from Zurek [20])

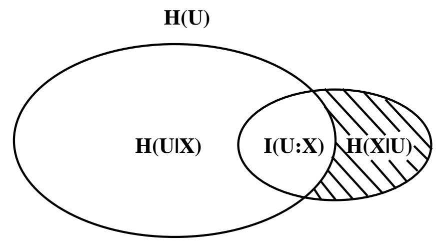

Figure 3: Diagram for a system X with entropy *not* shared with U (randomness).

position of the molecule) effects the extraction of useful work, in apparent violation of the second law. However, the box with piston is not a closed system. The information obtained via the measurement process has to be recorded, or equivalently, the information recorded in the previous cycle has to be replaced. Landauer [12] determined that erasing a bit of information is associated with the dissipation of $kT\log 2$ of heat, which turns out to be the precise amount of work gained in the Szilard cycle. Then, the second law is inviolate, as in fact implied by (3) when written as $H(U) = H(U|X) + I(U:X)$[5].

This famous example illustrates two messages: the thermodynamics of the measurement process is consistent if information is treated properly, and any information gained via a reversible measurement can be used to *reduce* the (conditional) entropy of the "universe". There is thus *no* "valueless" information, as by its definition (3) it can *always* be used to reduce the entropy of the system it was extracted from. Let us suppose that part of the entropy of X is *not* shared by U, as depicted in Fig. 3. Then, part of $H(X)$ does not convey any information about U, i.e., $I(U:X) < H(X)$. The randomness in X thus just corresponds to $H(X|U)$, the shaded area of $H(X)$ in Fig. 3. Clearly, this part is "valueless", i.e., it is *not* information, as it cannot be obtained from U via a measurement. Now, we can state our earlier definition of physical randomness precisely: a random string is one that cannot be used to reduce the entropy of the closed system under consideration, i.e., one that shares no entropy with closed system. Note that the definition of randomness is *conditional* on a closed system, or "universe", U. As a consequence we arrive at the intuitively correct notion that the determination of the randomness of a string depends on complete knowledge of the universe it is a part of, and may thus be undecidable if the universe is infinite (from an automatathory point of view, this is a reformulation of the famous "halting-problem" [19]). Conversely, a string may be random in one system while extremely "meaningful" in another[6]. Physical randomness of a string is thus the lack of information about its universe. Consequently we would like to identify the *physical complexity* of a string with *the amount of information about its universe*. Such a definition can be made precise, by constructing an automaton that realizes the *duality* between the measurement process and computation.

4 Physical Complexity

We define: A *Landauer-Turing (LT) machine* is a conventional but reversible Turing machine[7] supplemented with a finite but arbitrarily long tape u, in contact with a heat bath at temperature T. The tape u represents the physical universe of the LT-machine, absent in Kolmogorov's and Chaitin's construction. Its presence guarantees the duality mentioned above. The reversibility of the LT machine can be assured if none of the automaton rules (which govern the state of the read/write head and the reading and writing of bits as a function of the state of the machine) specify the writing of a symbol (0 or 1) *independently* of the state of the automaton. Then, any string computed by the LT-machine from tape u actually corresponds to a *measurement* performed on the (abstract) universe u, specifically, any string obtained by computation is necessarily either a part of u, or obtainable from u using universal rules. Moreover, a random string can not be computed by the LT-machine by construction. Conversely, any string s that *could* have been obtained by computation on the Landauer-Turing machine can be used to reduce the entropy of u (by reversing the computation). In the language of Bennett [5, 6], any string that can be used to decrease the entropy of the universe has

[5]This appears to be the consistent description of the measurement process in terms of a "physical entropy" that was intended by Zurek [20]. Thus, it turns out that Zurek's "physical entropy" is just the entropy of the universe, while the "remaining ignorance" is just the *conditional* entropy, which is the entropy of U as it appears to X.

[6]In Jorge Luis Borges' story "The Library of Babel" (quoted by Bennett in [6]), the librarians are faced with the task to decide which of the books of the library, which has a copy of all book-length sequences of letters, have meaning. Clearly, this determination will depend on the language ("universe") considered: a book written in Swahili, for example, may appear random to the librarians, but not to East Africans.

[7]Reversible Turing machines (or Bennett-Fredkin-Turing machines, based on conservative logic) have been shown to also be universal [3, 10].

"fuel value" equal to the heat dissipated when recording it.

We conclude also that algorithmically irregular strings (random strings in the nomenclature of Kolmogorov and Chaitin) are *physically non-random*, as they are the result of a computation. In fact, the proof for algorithmical randomness involves including the random string r *verbatim* in the program p (since r can not be the result of an algorithmic computation). Thence, according to Kolmogorov and Chaitin, it follows that $K(r) \geq |r|$ simply because $|p| \geq |r|$. However, to obtain r from p the Turing machine engages in *copying* the string r. Thus, if p is regarded as part of the universe u (as implied in the construction of the Landauer-Turing machine), writing r results in a universe containing *two* copies of r, and the entropy is lowered. Thus, r has fuel value.

Having determined the criterion for physical randomness, let us proceed to define the physical complexity $K_L(s:u)$ of a string s in universe u:

$$K_L(s:u) = \min\left(|p|:\ s = C_L(p,u)\right)\ , \tag{4}$$

where $C_L(p,u)$ is the result of running program p on the Landauer-Turing machine L supplied with universe u[8]. This construction ensures that $K_L(s:u)$ is equal to the number of bits in s which represent information in u. This quantity has an exact analog in the information theory of the measurement process. From this point of view, $K_L(s:u)$ represents the *mutual information* between s and the universe u, i.e., the amount of information gained about the universe from "measuring" s. Accordingly,

$$K_L(s:u) = I(s:u) = H(u) - H(u|s)\ . \tag{5}$$

Despite the formal similarity to Kolmogorov algorithmic *mutual* complexity [20], $K_L(s:u)$ is *not* a difference of the physical complexities of u and $u|s$, as physical complexities are *only* defined with reference to an environment. In other words, the physical complexity of s or u itself is undefined.

In practice, the quantity $I(s:u)$ is hard to obtain, mainly because its calculation either involves $H(u)$, the "entropy of the universe", which by virtue of its definition is unmeasurable (as you cannot observe a closed system). On the other hand, we can write (by Bayes' formula)

$$I(s:u) = H(s) - H(s|u)\ . \tag{6}$$

These quantities are also difficult to measure in most situations due to the finite size of the statistical ensembles. Thus, while the mutual information is not necessarily a practical measure, its automata-theoretic equivalent $K_L(s:u)$ on the other hand, potentially *is*. To see this, we would like to present two examples where an *approximate* determination of physical complexity is feasible, because Landauer-Turing machines are at hand.

[8] Note that technically, a Turing machine has only *one* tape. The subdivison of the tape of the Landauer-Turing machine into universe, program, and output tape is thus arbitrary. In fact, segments u, p, and s correspond to the subdivision of entropies into mutual and conditional entropies as in Figs. 1 or 3.

5 Complexity of Genomes

The first is certainly the more important example, as it concerns the determination of the complexity of segments of RNA. Our purpose here is to outline the practical aspects of such a determination based on the complexity measure proposed, rather than an examination of the feasibility of this method with current technology. Unfolded RNA can be viewed as a bit-string $s_{\rm RNA}$ which corresponds bit by bit to the DNA "program" that gave rise to it. In order to estimate the physical complexity of such a string, we need only to search for the shortest string (in bits) which generates the specific conformation displayed by $s_{\rm RNA}$.

Let us consider for the purpose of illustration the well-known molecule transfer RNA (tRNA). tRNA consists out of 76 nucleotides that contort into the well-known clover-leaf structure. For tRNA obtained from a given species, data can be collected which allows a classification of each position as either fixed, or volatile. In *Bacillus subtilis* for example, 21 of 52 binary reference positions[9] are constant, or "cold" in the thermodynamic picture, while 31 positions are variable ("hot") [9]. We are now looking for the shortest string which could produce this structure. Certainly, we have to count those sites that are fixed as information that is shared with the "universe" the string has evolved in. On the other hand, volatile sites carry no information about the "universe" by our definition: they are thermodynamically random. In order to determine the physical complexity of this string, we still need to assess whether the string composed of the fixed sites only is in fact the shortest that can be obtained. This of course cannot be answered with certainty, and this shortcoming is obviously related to the halting problem: in order to determine this with certainty, the entire universe would have to be known. Nevertheless, for strings that have emerged from billions of years of evolution. we may assume that nature has compactified the information to close to its maximum (not counting variable sites!). The physical complexity of the tRNA portion of *B. subtilis* would thus be 21 bits (of 52). While this number is not useful in itself, it should be possible to use it in comparison with the physical complexity of other strings of RNA.

Another example is an artificial one, yet conveys the flavor of the practical possibilities (and limitations) of

[9] In this analysis, nucleotides were classified as either purine (A or G) or pyrimidine (U or C). Counting base pairs as one position and excluding the anticodon region reduces the number of reference positions from 76 to 52.

physical complexity. Artificial living systems have been created [15, 1] which involve segments of (computer)-code self-replicating in a noisy environment replete with information. In such systems, the information in the environment (specified by the user) is transferred into the code *stochastically*. More precisely, the "accidental" discovery (via random mutations) of a sequence that benefits the string is "frozen" in the genome owing to the higher replication rate of its bearer. The replication of each string is effected by executing its code on a virtual computer, the analog of the Landauer-Turing machine. As the population equilibrates, the information-bearing sections of the code become apparent as they are *fixed*, while the volatile positions provide for genomic diversity without storage of information. Note that the determination of volatility of a site is only possible *statistically*, i.e., by examining ensembles of members of the same "species". The physical complexity of a species of strings thus corresponds to the number of non-volatile instructions in the code within a given environment. As emphasized throughout this paper, the information-content of a genomic string *by itself* (without referring to an environment) is a meaningless concept. In such artificial living systems, the increase of physical complexity, which coincides with increasing acquisition and storage of information, can be monitored directly [2], which illustrates the usefulness of this measure.

6 Conclusions

One of the more frequent intuitive characterizations of complexity that is often heard takes the form: "The complexity of X must be related to the amount of information necessary to describe X...". This is a fallacy, because information is *not* description, but rather mutual entropy: only that part of the description of X is information which is correlated (has "a meaning in") a given "universe" U that X is in contact with. If information is thus properly defined, we observe that complexity simply equals information. A random string (one that has no mutual entropy with a given U), then, has zero complexity (with respect to this U). This is in fact how we *define* randomness. This notion of physical complexity of a string, based on the thermodynamics of measurement, can be obtained equivalently from automata theory as the length of the smallest program that computes the string from the "universe" tape of the Landauer-Turing machine. The complexity of a string that *appears* random is a priori *undecidable*, due to the fundamental halting problem. In general, a string appears random up to the point where a computation is found that produces the string, or where this string is found *verbatim* in the Landauer-Turing "universe". Finally, it is the parallelism between computation and measurement that allows for practical estimates of the complexity of bit-strings or genomes, as those bits or nucleotides that are fixed in a statistical ensemble.

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